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LETTER TO THE EDITOR

A symmetry in the finite-temperature Casimir effect

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Abstract. Vacuum fluctuations at finite temperature between two plane walls give rise to a Casimir energy which has a simple symmetry between high and low temperatures. This symmetry is most easily understood in a derivation based on functional methods using dimensional regularisation and generalised zeta functions.

The Casimir effect is due to the vacuum fluctuations of quantum fields. Each mode of a bosonic field contributes $+\frac{1}{2}\hbar\omega$ to the vacuum energy while a fermionic mode contributes $-\frac{1}{2}\hbar\omega$. Casimir [1] was able to sum these mode contributions for an electromagnetic field between two parallel plates with separation L at zero temperature. He found an attractive force given by the negative pressure $P=-\pi^2/240L^4$ in units where $\hbar=c=1$.

More recently, Johnson [2] calculated the corresponding effect due to a massless Dirac field confined between two parallel plates by the MIT boundary condition. The pressure is now found to be $P=-7\pi^2/960L^4$ which is $\frac{7}{4}$ of the electromagnetic pressure. This ratio is exactly the same as one has between the corresponding two pressures at very high temperatures where the presence of the boundaries is no longer felt and the vacuum fluctuations are just free black-body radiation.

The equality of the ratios of the fermionic and bosonic Casimir forces at very low and high temperatures, T, is no accident. It comes about due to a symmetry in the free energy density $F(T, L)$ of a massless field between two parallel plates. For the electromagnetic field it was first noticed by Brown and Maclay [3]. They showed that the dimensionless function $f(\xi)=L^3F(T, L)$, where $\xi=LT$ is also dimensionless in units where Boltzmann's constant $k=1$, satisfies

$$f(\xi)=(2\xi)^4 f(1/4\xi) \tag{1}$$

which is a symmetry between high and low temperatures. Actually, the function $f(\xi)$ as defined by Brown and Maclay [3] is only a part of the free energy, but the complete free energy also has this symmetry. Recently Gundersen and Ravndal [4] noticed that the same symmetry also obtains for the scaled free energy of a fermionic field between parallel MIT plates.

We clarify these results by giving a simple derivation using functional methods which are ideally suited for this problem. In this formulation the origin of the symmetry

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between the high- and low-temperature results is manifest and the observed ratio $\frac{7}{4}$ between the fermionic and bosonic Casimir forces at both very low and very high temperatures is thereby explained.

The free energy for a massless field at temperature $T=\beta^{-1}$ is obtained by evaluating the partition function on a manifold where the time direction is Euclideanised and compactified to a circle (S^1) of size β. In addition, the Casimir geometry corresponds to compactifying one of the spatial directions to size L, the distance between the parallel plates.

Bose fields are periodic on S^1 and satisfy spatial boundary conditions of the Dirichlet or Neumann type, depending on the physics of the system being investigated. Fermi fields, on the other hand, are antiperiodic on S^1 and are usually constrained spatially by the MIT boundary conditions.

Consider now a free massless Dirac field, whose partition function

$$\mathscr{Z}=\int \mathscr{D}\psi\mathscr{D}\bar{\psi}\exp\left(-\int_0^\beta \mathrm{d}x\,\bar{\psi}\not{D}\psi\right) \tag{2}$$

is Gaussian and therefore evaluates in four dimensions to [5]

$$\mathscr{Z}=\det(\not{D})=\det{}^{1/2}(\not{D}^2). \tag{3}$$

Thus the effective potential, or free energy density, is simply given by

$$\beta F=-\tfrac{1}{2}\mathrm{Tr}\ln(-D^2). \tag{4}$$

The trace is over all the non-zero eigenvalues of the differential operator acting on our partially compactified manifold where the spectrum is semicontinuous. With essentially antiperiodic boundary conditions in both compactified directions the discrete eigenvalues run over the half-integers [6] and we have (in the absence of any chiral angle in the MIT boundary condition) that

$$\beta F=-\sum_{n,m=-\infty}^{\infty}\int\frac{\mathrm{d}^2k_T}{(2\pi)^2}\ln[k_T^2+(\pi/2L)^2(2m+1)^2+(\pi/\beta)^2(2n+1)^2] \tag{5}$$

where k_T is the momentum parallel to the plates. It is obvious that the spectrum, and therefore the free energy F, reflects the symmetry of the spacetime manifold under investigation. By inspection, we see that it is invariant under the interchange of $2L$ and β. This is essentially the symmetry we want to elucidate.

To obtain the functional form of F we employ the useful result [7]

$$\int\frac{\mathrm{d}^dk}{(2\pi)^d}\ln(k^2+m^2)=-\frac{\Gamma(-\frac{1}{2}d)}{(4\pi)^{d/2}}m^d. \tag{6}$$

When applied to (5) after analytically continuing the number of transverse dimensions away from 2, this gives

$$\beta F=\lim_{d\to 2}\frac{\Gamma(-\frac{1}{2}d)}{(4\pi)^{d/2}}\left(\frac{\pi}{2L}\right)^d S(2\xi) \tag{7}$$

with

$$S(x)=\sum_{n,m=-\infty}^{\infty}[(2m+1)^2+x^2(2n+1)^2]^{d/2}. \tag{8}$$

This double sum can be written in terms of Epstein zeta functions which, for sufficiently large (real) argument s, is given by the series expansion [8]:

$$Z_2(a, b; s) = \sum_{n,m=-\infty}^{\infty}{}' [(ma)^2 + (nb)^2]^{-s} \tag{9}$$

where the sum extends over all positive and negative integers except $m = n = 0$. We can then write

$$S(x) = (1 + 2^d)Z_2(1, x; -d/2) - Z_2(1, 2x; -d/2) - Z_2(2, x; -d/2). \tag{10}$$

The analytic continuation of (7) to its physical value is now easily done by using the reflection formula [8] satisfied by the zeta function:

$$\Gamma(s)\pi^{-s}Z_2(a, b; s) = (ab)^{-1}\Gamma(1-s)\pi^{s-1}Z_2(a^{-1}, b^{-1}; 1-s). \tag{11}$$

Combining (11) with (7) and (10), the dangerous gamma functions drop out and we find in the physical limit ($d \to 2$) the final result for the scaled free energy $f(\xi) = L^3F(T, L)$:

$$f(\xi) = \frac{(2\xi)^4}{8\pi^2} C(2\xi) \tag{12}$$

where

$$C(x) = \tfrac{5}{4}Z_2(1, x; 2) - 2Z_2(1, 2x; 2) - 2Z_2(2, x; 2) \tag{13}$$

is a finite function which can be written as

$$C(x) = \sum_{n,m=-\infty}^{\infty}{}' (-1)^{m+n}[m^2 + (xn)^2]^{-2} \tag{14}$$

where the point $m = n = 0$ is again excluded from the summation.

For an electromagnetic field trapped between reflecting plates a result almost identical to (12) is known [3]. The only difference is that the doubly periodic boundary conditions appropriate in this case avoid the alternating sign in the representation of $C(2\xi)$ given by (14). It is therefore simply replaced by the Epstein zeta function $Z_2(1, 2\xi; 2)$. This will also give the free energy of two massless scalar fields between the plates, satisfying boundary conditions of, respectively, the Dirichlet and Neumann type [9].

We see that, if we hold L fixed, (1) tells us that the free energy of these massless fields between parallel plates at low temperature is given by the free energy at high temperature. Hence the free energy at $T = 0$, which is the Casimir energy, is completely determined by the Stefan-Boltzmann law for black-body radiation. Since it is well known that the free energy of hot electrons differs by $\frac{7}{4}$ from the free energy of photons at the same temperature, and (1) is satisfied for both species of particles, the same ratio is also valid for the vacuum energies of these fields between the plates at zero temperature.

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